

TIME AND BAND LIMITING FOR MATRIX VALUED FUNCTIONS: AN INTEGRAL AND A COMMUTING DIFFERENTIAL OPERATOR

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ABSTRACT. The problem of recovering a signal of finite duration from a piece of its Fourier transform was solved at Bell Labs in the 1960's, by exploiting a “miracle”: a certain naturally appearing integral operator commutes with an explicit differential one. Here we show that this same miracle holds in a matrix valued version of the same problem.

1. INTRODUCTION

A typical inverse problem with partial and noisy data can be formulated as follows:

The unknown is a function $f(x)$ defined in physical space, and we may have some a priori knowledge about $f(x)$. For instance f could be supported in a compact part of physical space. The data is a piece of its Fourier transform $\mathcal{F}(k)$ contaminated by noise. What is the best use one can make of this data to get a stable reconstruction of $f(x)$?

Many imaging problems fall under this umbrella, and maybe the question above was put in this precise form first by Claude Shannon while coming up with a mathematical foundation for Communication Theory back around 1950. His problem was addressed in a remarkable series of papers by (different combinations of) D. Slepian, H. Landau and H. Pollak all working at Bell Labs where C. Shannon had done initial work before leaving for MIT. See [SP61, LP61, LP62, Sle64, LP78].

One of us run into a situation of this kind in dealing with “limited angle tomography,” see [DG79, DG81, Nat01]. A detailed exploration of the corresponding “singular value decomposition” for this problem, see [Dav83, Lou86, Grü86], made strong contact with the work of these workers at Bell Labs.

With this motivation at hand, we can give an account of what we do in this paper: we start with a (matrix valued) version of a second order differential operator. Here Shannon would have started with the (scalar valued) second derivative. Our harmonic analysis in terms of eigenfunctions of this operator is the analog of Fourier analysis in the classical case.

We then build the analog of the “time-and-band limiting” integral operator, which we will denote by S . We then show that the same “lucky accident” found by the workers at Bell labs holds here

Date: January 20, 2017.

2010 *Mathematics Subject Classification.* 33C45, 22E45, 33C47.

Key words and phrases. Time-band limiting, Double concentration, Matrix valued orthogonal polynomials.

This research was supported in part by CONICET grant PIP 112-200801-01533, SeCyT-UNC and FONDECYT 3160646.

too: we can exhibit a second order differential operator, denoted by \tilde{D} , such that

$$S\tilde{D} = \tilde{D}S.$$

This has, as in the original case of Shannon, very important numerical consequences: it gives a reliable way to compute the eigenvectors of S , something that cannot be done otherwise.

The eigenfunctions of S and \tilde{D} are the same ones, but by using the differential operator instead of the integral one we have a manageable numerical problem: while the integral operator has a spectrum with eigenvalues that are extremely close together, the differential one has a very spread out spectrum, resulting in a stable numerical computation.

Previous explorations of a commutativity property similar to the one above in the matrix valued case can be seen in [GPZ15, CG15], dealing with a full matrix and a narrow banded one. Our work here is related to that in [GPZ15] but there are some important differences: instead of studying the commutant of a matrix M , which arises in the problem of time and band limiting, we look for a symmetric differential operators of order two commuting with an integral operator, which is the other operator of interest featuring in the time and band limiting problem. This extends the work started in the previous references and is much closer to the early work at Bell Labs in the scalar case. When they consider the unit circle as physical space, they also have to deal with two different situations, as we do.

The work [GLP82] was written with no particular application in mind but some years later some applications were developed from the results thereof, see [SD06, SDW06] and its references. We believe that the results obtained here, where one is dealing with matrix valued functions defined on spheres, will open the possibility of new applications in the future, most likely to an inverse problem involving tensor quantities. While we deal here with 2×2 matrix valued functions, the theory of matrix valued spherical functions yields situations with matrices of arbitrary size. For a sample of references on “tensor tomography” see [GRZ⁺99, PSU14] and references in these papers.

We hope that by placing these mathematical tools in front of the inverse problem community we may encourage people to use them in concrete imaging problems.

2. PRELIMINARIES

We follow the same notation as in [GPZ15], but we include some motivation here for benefit of the reader. Let $W(x)$ be a matrix weight function in the open interval (a, b) and let $\{Q_w(x)\}_{w \in \mathbb{N}_0}$ be a sequence of matrix orthonormal polynomials with respect to the weight $W(x)$.

The Hilbert spaces $\ell^2(M_R, \mathbb{N}_0)$ and $L^2((a, b), W(t)dt)$ are given by the real valued $R \times R$ matrix sequences $(C_w)_{w \in \mathbb{N}_0}$ such that $\sum_{w=0}^{\infty} \text{tr}(C_w C_w^*) < \infty$ and all measurable matrix valued functions $f(x)$, $x \in (a, b)$, satisfying $\int_a^b \text{tr}(f(x)W(x)f^*(x)) dx < \infty$, respectively. A natural analog of the Fourier transform is the isometry $F : \ell^2(M_R, \mathbb{N}_0) \longrightarrow L^2(W)$ given by

$$(C_w)_{w=0}^{\infty} \longmapsto \sum_{w=0}^{\infty} C_w Q_w(x).$$

In our case, the matrix-polynomials are dense in $L^2(W)$, then this map is unitary with the inverse $F^{-1} : L^2(W) \rightarrow \ell^2(M_R, \mathbb{N}_0)$ given by

$$f \mapsto C_w = \int_a^b f(x) W(x) Q_w^*(x) dx.$$

If we consider the problem of determining a function f , from the following (typically noisy) data: f has support on the compact set $[0, N]$ and its Fourier transform Ff is known on a compact set $[a, \alpha]$, one concludes that we need to compute the singular vectors (and values) of the operator $E : \ell^2(M_R, \mathbb{N}_0) \rightarrow L^2(W)$ given by

$$Ef = \chi_\alpha F \chi_N f,$$

where χ_N is the *time limiting operator* and χ_α is the *band limiting operator*.

At level N , χ_N acts on $\ell^2(M_R, \mathbb{N}_0)$ by simply setting equal to zero all the components with index larger than N . At level α , χ_α acts on $L^2(W)$ by multiplication by the characteristic function of the interval (a, α) , $a < \alpha \leq b$.

This leads us to study the eigenvectors of the operators

$$E^*E = \chi_N F^{-1} \chi_\alpha F \chi_N \quad \text{and} \quad EE^* = \chi_\alpha F \chi_N F^{-1} \chi_\alpha.$$

The operator E^*E , acting on $\ell^2(M_R, \mathbb{N}_0)$ is just a finite dimensional block-matrix M , and each block is given by

$$(M)_{m,n} = (E^*E)_{m,n} = \int_a^\alpha Q_m(x) W(x) Q_n^*(x) dx, \quad 0 \leq m, n \leq N.$$

The second operator $S = EE^*$ acts on $L^2((a, \alpha), W(t)dt)$ by means of the integral kernel

$$(1) \quad k(x, y) = \sum_{w=0}^N Q_w^*(x) Q_w(y).$$

The action of $k(x, y)$ is spelled out later in formula (2) more explicitly.

For general N and α there is no hope of finding the eigenfunctions of EE^* and E^*E analytically. However, there is a strategy to solve this typical inverse problem: finding an operator with simple spectrum which would have the same eigenfunctions as the operators EE^* or E^*E . This is exactly what Slepian, Landau and Pollack did in the scalar case, when dealing with the unit circle and the usual Fourier analysis. They discovered the following properties:

- For each N, α there exists a symmetric tridiagonal matrix L , with simple spectrum, commuting with M .
- For each N, α there exists a self-adjoint differential operator D , with simple spectrum, commuting with the integral operator $S = EE^*$.

More than this is true: thanks to this lucky accident one replaces a costly ill-conditioned problem by a manageable well-conditioned one. It is hard to ask for a better situation.

In [GPZ15] we have dealt with the analog of the first property for the case discussed in [PZ16]. In this paper we address the second one of the properties above in the same situation.

In both cases, the role of the unit circle will be taken up by the n -dimensional sphere. We consider 2×2 matrix valued functions defined on the sphere with the appropriate invariance that makes them functions of the colatitude θ and we use $x = \cos(\theta)$ as the variable. The role of the

Fourier transform is taken by the expansion of our functions in terms of a basis of matrix valued orthogonal polynomials described below. This is similar to the situation discussed in [GLP82] except for the crucial fact that our functions are now matrix valued.

3. INTEGRAL AND DIFFERENTIAL OPERATORS

Given the sequence of matrix orthonormal polynomials $\{Q_w\}_{w \geq 0}$ with respect to the weight W , we fix a natural number N and $\alpha \in (-1, 1)$. We consider the integral operator S with kernel k , defined in (1), acting on $L^2((-1, \alpha), W)$ “from the right hand side”:

$$(2) \quad (fS)(x) = \int_{-1}^{\alpha} f(y)W(y)(k(x, y))^* dy$$

The restriction to the interval $[-1, \alpha]$ implements “band-limiting” while the restriction to the range $0, 1, \dots, N$ takes care of “time-limiting.” In the language of [GLP82] where we were dealing with scalar valued functions defined on spheres the first restriction gives us a “spherical cap” while the second one amounts to truncating the expansion in spherical harmonics.

The aim of this paper is to prove that there exists a symmetric differential operator \tilde{D} , defined in $[-1, \alpha]$, commuting with the integral operator S , that is

$$\tilde{D}S = S\tilde{D}.$$

Symmetry for the differential operator \tilde{D} means that

$$\langle P\tilde{D}, Q \rangle_{\alpha} = \langle P, Q\tilde{D} \rangle_{\alpha},$$

for an appropriate dense set of functions P, Q where

$$(3) \quad \langle P, Q \rangle_{\alpha} = \int_{-1}^{\alpha} P(x)W(x)Q^*(x) dx.$$

Notice that in principle there is no guarantee that we will find any such \tilde{D} except for a scalar multiple of the identity. For the problem at hand we need to exhibit a differential operator \tilde{D} that has a simple spectrum, which would imply that its eigenfunctions are also eigenfunctions of the integral operator S (see comments in Section 5).

Proposition 3.1. *Let C be a symmetric differential operator and S be an integral operator with kernel k . Then*

$$CS = SC \quad \text{if and only if} \quad (k(x, y)^*) C_x = (k(x, y) C_y)^*.$$

(C_x is meant to emphasize that C acts on the variable x).

Proof. Let us observe that from the symmetry of C we have

$$((fC)S)(x) = \int_{-1}^{\alpha} (fC_y)(y)W(y)(k(x, y))^* dy = \int_{-1}^{\alpha} f(y)W(y)(k(x, y)C_y)^* dy.$$

On the other hand

$$((fS)C)(x) = \int_{-1}^{\alpha} f(y)W(y)(k(x, y)^*) C_x dy.$$

Therefore the proposition follows easily. □

The polynomials considered here are those studied in [PZ16], given by the matrix-valued spherical functions associated with the n -dimensional sphere $S^n \simeq G/K$, with $(G, K) = (\mathrm{SO}(n+1), \mathrm{SO}(n))$, studied in [TZ14]. These spherical functions give rise to sequences $\{R_w\}_{w \in \mathbb{N}_0}$ of monic matrix orthogonal polynomials depending on two parameters n and p in \mathbb{R} such that $0 < p < n$. Namely,

$$R_w(x) = \frac{w!(n+1)}{2^w \left(\frac{n+1}{2}\right)_w} \begin{pmatrix} \frac{1}{n+1} C_w^{\frac{n+1}{2}}(x) + \frac{1}{p+w} C_{w-2}^{\frac{n+3}{2}}(x) & \frac{1}{p+w} C_{w-1}^{\frac{n+3}{2}}(x) \\ \frac{1}{n-p+w} C_{w-1}^{\frac{n+3}{2}}(x) & \frac{1}{n+1} C_w^{\frac{n+1}{2}}(x) + \frac{1}{n-p+w} C_{w-2}^{\frac{n+3}{2}}(x) \end{pmatrix},$$

where $C_w^\lambda(x)$ denotes the w -th Gegenbauer polynomial. The matrix polynomials $\{R_w\}_{w \geq 0}$ are orthogonal with respect to the weight matrix

$$(4) \quad W(x) = W_{p,n}(x) = (1-x^2)^{\frac{n}{2}-1} \begin{pmatrix} px^2 + n-p & -nx \\ -nx & (n-p)x^2 + p \end{pmatrix}, \quad x \in [-1, 1].$$

The kernel $k(x, y)$ appearing in the definition of the integral operator S is given in terms of the orthonormal sequence of matrix polynomials

$$(5) \quad Q_w = S_w R_w,$$

where $S_w = \|R_w\|^{-1}$ is the inverse of the matrix norm of R_w given by

$$(6) \quad \langle R_w, R_w \rangle = \|R_w\|^2 = \frac{\sqrt{\pi} \Gamma(\frac{n}{2} + 1)(n+1)_w}{w!(n+1)(n+2w+1)\Gamma(\frac{n}{2} + \frac{3}{2})} \begin{pmatrix} \frac{p(n-p+w+1)}{p+w} & 0 \\ 0 & \frac{(n-p)(p+w+1)}{n-p+w} \end{pmatrix}.$$

Theorem 3.2. (*Theorem 3.1*, [PZ16].) *For each $w \in \mathbb{N}_0$, the matrix polynomial R_w satisfies $R_w D = \Lambda_w R_w(x)$, where*

$$D = \frac{d^2}{dx^2} (1-x^2) - \frac{d}{dx} \left((n+2)x + 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) - \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix},$$

and the eigenvalue is given by

$$\Lambda_w(D) = \begin{pmatrix} -w(w+n+1)-p & 0 \\ 0 & -w(w+n+1)-n+p \end{pmatrix}.$$

Let us remark that this differential operator acts on the variable x “from the right hand side.”

Let us introduce the following right hand side differential operator \tilde{D} , acting on nice functions in the interval $(-1, \alpha)$.

$$(7) \quad \begin{aligned} \tilde{D} = & \frac{d^2}{dx^2} (x^2 - 1)(x - \alpha) + \frac{d}{dx} \left((n+3)x^2 - \alpha(n+2)x - 1 + 2(x - \alpha) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \\ & + \left(-N(N+n+2)x + \alpha(n-2p) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & n-p+1 \\ p+1 & 0 \end{pmatrix} \right). \end{aligned}$$

We will prove later that \tilde{D} is a symmetric operator and that it satisfies $(k(x, y)^*) \tilde{D}_x = (k(x, y) \tilde{D}_y)^*$. Therefore \tilde{D} commutes with the integral operator S by Proposition 3.1.

Let us observe that the differential operator \tilde{D} is somehow related to the differential operator D , given in Theorem 3.2. Explicitly, we have

$$\tilde{D} = -D(x - \alpha) + \frac{d}{dx}(x^2 - 1) + \left(-N(N + n + 2) - \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix}\right)x + \alpha(n - p) + \begin{pmatrix} 0 & n-p+1 \\ p+1 & 0 \end{pmatrix}.$$

This relation between the analogs of D and \tilde{D} is simpler in the case worked out at Bell Labs. For simplicity, we use the notation

$$E_1 = -N(N + n + 2)\mathbf{I} - \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix}, \quad E_0 = \begin{pmatrix} 0 & n-p+1 \\ p+1 & 0 \end{pmatrix}.$$

Theorem 3.3. *The differential operator \tilde{D} is symmetric with respect to the matrix valued inner product $\langle \cdot, \cdot \rangle_\alpha$ given in (3).*

Proof. From [GPT03] or [DG04] we have that a differential operator $D = \frac{d^2}{dx^2}F_2(x) + \frac{d}{dx}F_1(x) + F_0$ is symmetric with respect to a weight W defined in (a, b) if and only if it satisfies, for $a < x < b$, the symmetry equations

$$\begin{aligned} (8) \quad & F_2W = WF_2^*, \\ & 2(F_2W)' - F_1W = WF_1^*, \\ & (F_2W)'' - (F_1W)' + F_0W = WF_0^* \end{aligned}$$

and the boundary conditions

$$(9) \quad \lim_{x \rightarrow a, b} F_2(x)W(x) = 0, \quad \lim_{x \rightarrow a, b} (F_1(x)W(x) - WF_1^*(x)) = 0.$$

We have the following relations among the coefficients of the differential operators D and \tilde{D} ,

$$\begin{aligned} \tilde{F}_2 &= -(x - \alpha)F_2, \\ \tilde{F}_1 &= -(x - \alpha)F_1 + (x^2 - 1)I, \\ \tilde{F}_0 &= -(x - \alpha)F_0 + xE_1 + E_0 + \alpha I. \end{aligned}$$

We know that D is a symmetric operator with respect to the weight W for a subspace of functions defined in the interval $(-1, 1)$, thus, the pair $\{W, D\}$ satisfies the equations (8) and (9) with $a = -1, b = 1$. From this and by straightforward computations we verify that \tilde{D} and W also satisfy the equations (8) and the boundary conditions (9) with $a = -1, b = \alpha$. \square

To prove that the differential operator \tilde{D} commutes with the integral operator S we have to verify that $(k(x, y)^*)\tilde{D}_x = (k(x, y)\tilde{D}_y)^*$, see Proposition 3.1. For this purpose, we will need three different things: the explicit expressions of the three term recursion relation, the Christoffel-Darboux formula and a differentiation formula for a matrix orthonormal sequence $\{Q_w\}_{w \geq 0}$. For the benefit of the reader we postpone the statements and proofs to Section 4, and proceed directly to the statement of our main result.

The three term recursion mentioned above can be thought of as a difference operator acting on the w variable “from the left hand side.” We are in the presence of a “bispectral” situation, and we will eventually see that it gives us the “commuting miracle” mentioned at the beginning of this paper.

Theorem 3.4. *The differential operator \tilde{D} satisfies*

$$(k(x, y)^*) \tilde{D}_x = (k(x, y) \tilde{D}_y)^*.$$

Proof. Let D the right hand side differential operator introduced in Theorem 3.2. We have that the orthonormal polynomials $\{Q_w\}_w$ are eigenfunctions of D with the eigenvalue Λ_w , i.e. $Q_w D = \Lambda_w Q_w$. Then

$$\begin{aligned} (k(x, y) \tilde{D}_y)^* &= -(y - \alpha) \sum_{w=0}^N Q_w^*(y) \Lambda_w Q_w(x) + (y^2 - 1) \sum_{w=0}^N \frac{d}{dy} Q_w^*(y) Q_w(x) \\ &\quad + \sum_{w=0}^N (y E_1 + E_0 + \alpha(n - p))^* Q_w^*(y) Q_w(x), \end{aligned}$$

and similarly

$$\begin{aligned} (k(x, y)^*) \tilde{D}_x &= -(x - \alpha) \sum_{w=0}^N Q_w^*(y) \Lambda_w Q_w(x) + (x^2 - 1) \sum_{w=0}^N \frac{d}{dx} Q_w^*(y) Q_w(x) \\ &\quad + \sum_{w=0}^N Q_w^*(y) Q_w(x) (x E_1 + E_0 + \alpha(n - p)). \end{aligned}$$

Thus,

$$\begin{aligned} (10) \quad (k \tilde{D}_y)^* - (k^* \tilde{D}_x) &= \sum_{w=0}^N \left((x - y) Q_w^*(y) \Lambda_w Q_w(x) + (y^2 - 1) \frac{d}{dy} Q_w^*(y) Q_w(x) \right. \\ &\quad \left. - (x^2 - 1) Q_w^*(y) \frac{d}{dx} Q_w(x) + (E_0^* + y E_1^*) Q_w^*(y) Q_w(x) - Q_w^*(y) Q_w(x) (E_0 + x E_1) \right). \end{aligned}$$

By using the differentiation formula given in Proposition 4.5, we have

$$\begin{aligned} (11) \quad (k \tilde{D}_y)^* - (k^* \tilde{D}_x) &= \sum_{w=0}^N (x - y) Q_w^*(y) \Lambda_w Q_w(x) + y Q_w^*(y) (w - \overline{F}_w^*) Q_w(x) \\ &\quad - x Q_w^*(y) (w - \overline{F}_w) Q_w(x) + Q_w^*(y) (\overline{G}_w - \overline{G}_w^*) Q_w(x) \\ &\quad + (E_0 + y(F_w + E_1) - \tilde{G}_w)^* Q_w^*(y) Q_w(x) - Q_w^*(y) Q_w(x) (E_0 + x(F_w + E_1) - \tilde{G}_w) \\ &\quad - Q_{w-1}^*(y) \overline{H}_w^* Q_w(x) + Q_w^*(y) \overline{H}_w Q_{w-1}(x). \end{aligned}$$

If we take

$$a_{21} = -1 - \frac{n+2w}{(p+w)(n-p+w)} \quad \text{and} \quad a_{22} = c_{22} = 0,$$

in the matrices given in Propositions 4.2 and 4.5, we have

$$\begin{aligned} F_w &= \begin{pmatrix} p & 0 \\ 0 & n - p \end{pmatrix}, & G_w &= - \begin{pmatrix} 0 & \frac{p(n-p+w+1)}{p+w} \\ \frac{(n-p)(p+w+1)}{n=p+w} & 0 \end{pmatrix}, \\ \tilde{G}_w &= \begin{pmatrix} 0 & n - p + 1 \\ p + 1 & 0 \end{pmatrix}, & H_w &= (n + 2w + n + 1) A_w, \end{aligned}$$

where A_w is the matrix in the three term recursion relation given in (13). In particular we have

$$F_w + E_1 = -N(N + n + 2)\mathbf{I} \quad \text{and} \quad \tilde{G}_w = E_0.$$

By using the explicit expression of $\|R_w\|$ (see (6)), we get

$$\overline{F}_w = \|R_w\|^{-1} F_w \|R_w\| = F_w = \overline{F}_w^*, \quad \overline{G}_w = \overline{G}_w^*, \quad \overline{H}_w = (n + 2w + n + 1)\tilde{A}_w.$$

Therefore from (11) we obtain

$$(12) \quad \begin{aligned} (k\tilde{D}_y)^* - (k^*\tilde{D}_x) &= \sum_{w=0}^N (x-y)Q_w^*(y)\Lambda_w Q_w(x) + (x-y)Q_w^*(y)(F_w - w)Q_w(x) \\ &\quad + (x-y)N(N+n+2)Q_w^*(y)Q_w(x) \\ &\quad - (n+2w+1)\left(Q_{w-1}^*(y)\tilde{A}_w^*Q_w(x) - Q_w^*(y)\tilde{A}_wQ_{w-1}(x)\right). \end{aligned}$$

By using the Christoffel-Darboux formula given in Proposition 4.1,

$$Q_{w-1}^*(y)\tilde{A}_w^*Q_w(x) - Q_w^*(y)\tilde{A}_wQ_{w-1}(x) = (x-y)\sum_{k=0}^{w-1}Q_k^*(y)Q_k(x),$$

and the fact that $\Lambda_w = \Lambda_w(D) = -w(w+n+1) - F_w$ (See Theorem 3.2) we have

$$\begin{aligned} \frac{(k\tilde{D}_y)^* - (k^*\tilde{D}_x)}{(x-y)} &= \\ &= \sum_{w=0}^N \left(Q_w^*(y)(-w(w+n+2) + N(N+n+2))Q_w(x) - (n+2w+1)\sum_{k=0}^{w-1}Q_k^*(y)Q_k(x) \right) \\ &= \sum_{w=0}^N (-w(w+n+2) + N(N+n+2))Q_w^*(y)Q_w(x) - \sum_{w=0}^{N-1} \left(\sum_{j=w+1}^N (n+2j+1) \right) Q_w^*(y)Q_w(x) \\ &= 0. \end{aligned}$$

Therefore we get $(k\tilde{D}_y)^* = (k^*\tilde{D}_x)$ and this concludes the proof of the theorem. \square

Corollary 3.5. *The differential operator \tilde{D} commutes with the integral operator S .*

4. PROPERTIES OF THE RELEVANT ORTHOGONAL POLYNOMIALS

In this section we give some results about the sequence of matrix orthogonal polynomials with respect to the weight $W(x)$, introduced in (4). Most of these results were used in the previous section to arrive at our main result.

4.1. Three term recursion relation. From the three term recursion relation for the sequence $\{P_w\}_{w \in \mathbb{N}_0}$ given in [PZ16] (Theorem 4.1), we get the three-term recursion relation for the monic orthogonal polynomials $\{R_w\}_{w \in \mathbb{N}_0}$

$$(13) \quad x R_w(x) = A_w R_{w-1}(x) + B_w R_w(x) + R_{w+1}(x),$$

where

$$A_w = \frac{w(n+w)}{(n+2w-1)(p+w)(n-p+w)(2w+n+1)} \begin{pmatrix} (p+w-1)(n-p+w+1) & 0 \\ 0 & (p+w+1)(n-p+w-1) \end{pmatrix},$$

$$B_w = \begin{pmatrix} 0 & \frac{-p}{(p+w)(p+w+1)} \\ \frac{-(n-p)}{(n-p+w)(n-p+w+1)} & 0 \end{pmatrix}.$$

We have

$$A_w = \|R_w\|^2 \|R_{w-1}\|^{-2}, \quad (B_w \|R_w\|^2)^* = B_w \|R_w\|^2.$$

In fact, from the three term recursion relation (13) we get

$$\begin{aligned} A_w \|R_{w-1}\|^2 &= \langle x R_w, R_{w-1} \rangle = \langle R_w, x R_{w-1} \rangle = \|R_w\|^2, \\ B_w \|R_w\|^2 &= \langle x R_w, R_w \rangle = \langle R_w, x R_w \rangle = \|R_w\|^2 B_w^*. \end{aligned}$$

An orthonormal sequence of matrix polynomials is given in terms of R_w by $Q_w = S_w R_w$, where $S_w = \|R_w\|^{-1}$ is the inverse of the matrix $\|R_w\|$. Therefore the orthonormal polynomials $\{Q_w\}_{w \in \mathbb{N}_0}$ satisfy the three-term recursion relation

$$x Q_w(x) = \tilde{A}_w Q_{w-1}(x) + \tilde{B}_w Q_w(x) + \tilde{A}_{w+1}^* Q_{w+1}(x),$$

where

$$\tilde{A}_w = S_w A_w S_{w-1}^{-1} = \|R_w\| \|R_{w-1}\|^{-1} \quad \text{and} \quad \tilde{B}_w = \|R_w\|^{-1} B_w \|R_w\|.$$

We observe that $\tilde{B}_w^* = \tilde{B}_w$.

Proposition 4.1. *The sequence of matrix orthonormal polynomials $\{Q_w\}_{w \geq 0}$ satisfies the following Christoffel-Darboux formula*

$$(14) \quad Q_{w-1}^*(y) \tilde{A}_w^* Q_w(x) - Q_w^*(y) \tilde{A}_w Q_{w-1}(x) = (x - y) \sum_{k=0}^{w-1} Q_k^*(y) Q_k(x).$$

where

$$\tilde{A}_w = S_w A_w S_{w-1}^{-1} = \|R_w\| \|R_{w-1}\|^{-1} \quad \text{and} \quad \tilde{B}_w = \|R_w\|^{-1} B_w \|R_w\|.$$

Proof. This result is proved in [Dur96]. □

4.2. Differentiation formulas. In this section we obtain several differentiation formulas for monic orthogonal polynomials. There are four free parameters, namely a_{21} , c_{12} , a_{11} and a_{22} .

Proposition 4.2. *Let $\{R_w\}$ be the monic orthogonal polynomials associated to the weight $W = W_{p,n}$ introduced in (4). We have*

$$(1-x^2)\frac{dR_w}{dx}(x) = -w x R_w(x) + x(F_w R_w(x) - R_w(x)F_w) + G_w R_w(x) \\ + R_w(x)\tilde{G}_w + H_w R_{w-1}(x),$$

where

$$F_w = -\frac{(n+2w)}{(p+w)(n-p+w)} \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix} - a_{21} \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix} + c_{12} \frac{(p+w)(n-p+w)}{(n-2p)} \begin{pmatrix} 0 & p \\ n-p & 0 \end{pmatrix} + a_{11} \text{Id},$$

$$G_w = \begin{pmatrix} 0 & \frac{p(n-p+w)}{(p+w)^2} \\ \frac{(n-p)(p+w)}{(n-p+w)^2} & 0 \end{pmatrix} + a_{21} \begin{pmatrix} 0 & \frac{p(n-p+w)}{(p+w)} \\ \frac{(n-p)(p+w)}{(n-p+w)} & 0 \end{pmatrix} \\ + c_{12}(w(w+n) - p(n-p)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{22} \text{Id},$$

$$\tilde{G}_w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \left(\frac{(n+2w)}{(p+w)(n-p+w)} + a_{21} \right) \begin{pmatrix} 0 & n-p \\ p & 0 \end{pmatrix} + c_{12} \begin{pmatrix} p(n-p) & 0 \\ 0 & -w(w+n) \end{pmatrix} - a_{22} \text{Id},$$

$$H_w = \frac{w(w+n)}{(p+w)(n-1+2w)(n-p+w)} \begin{pmatrix} (p+w-1)(n-p+w+1) & 0 \\ 0 & (p+w+1)(n-p+w-1) \end{pmatrix} \\ + c_{12} \frac{w(n+w)}{(n-1+2w)} \begin{pmatrix} 0 & \frac{p(n-p+w-1)}{p+w} \\ -\frac{(p+w-1)(n-p)}{(n-p+w)} & 0 \end{pmatrix}.$$

Proof. We look for constant matrices $F_w, \tilde{F}_w, G_w, \tilde{G}_w, H_w$ and \tilde{H}_w such that

$$(15) \quad (1-x^2)\frac{dR_w}{dx}(x) = xF_w R_w(x) + R_w(x)\tilde{F}_w + G_w R_w(x) + R_w(x)\tilde{G}_w + H_w R_{w-1}(x)$$

for all $w \in \mathbb{N}_0$. The polynomial R_w is of the form

$$R_w = \sum_{j=0}^w R_j^w x^j, \quad \text{with } R_w^w = \text{I}.$$

First of all, we can see that $-w = F_w + \tilde{F}_w$. Moreover, we have that (15) holds if and only if

$$(16) \quad (j+1)R_{j+1}^w - (j-1)R_{j-1}^w = -wR_{j-1}^w + F_w R_{j-1}^w - R_{j-1}^w F_w + G_w R_j^w + R_j^w \tilde{G}_w + H_w R_j^{w-1}.$$

The coefficients R_j^w were computed explicitly in [Zur16]. Depending on whether $w - j$ is odd or even, we have different expressions:

$$R_{w-2k}^w = \frac{w!(-1)^k}{2^{2k}k!(w-2k)!(\frac{n+1}{2}+w-k)_k} \begin{pmatrix} \frac{p+w-2k}{p+w} & 0 \\ 0 & \frac{n-p+w-2k}{n-p+w} \end{pmatrix}, \quad 0 \leq k \leq \lfloor w/2 \rfloor,$$

$$R_{w-2k-1}^w = \frac{w!(-1)^k}{2^{2k}k!(w-2k-1)!(\frac{n+1}{2}+w-k)_k} \begin{pmatrix} 0 & \frac{1}{p+w} \\ \frac{1}{n-p+w} & 0 \end{pmatrix}, \quad 0 \leq k \leq \lfloor (w-1)/2 \rfloor.$$

The equation (16) is equivalent to verifying the following identities, for every integer $k \geq 0$,

$$(w-2k+1)R_{w-2k+1}^w - (w-2k-1)R_{w-2k-1}^w$$

$$= -wR_{w-2k-1}^w + F_w R_{w-2k-1}^w - R_{w-2k-1}^w F_w + G_w R_{w-2k}^w + R_{w-2k}^w \tilde{G}_w + H_w R_{w-2k}^{w-1},$$

$$(w-2k)R_{w-2k}^w - (w-2k-2)R_{w-2k-2}^w$$

$$= -wR_{w-2k-2}^w + F_w R_{w-2k-2}^w - R_{w-2k-2}^w F_w + G_w R_{w-2k-1}^w + R_{w-2k-1}^w \tilde{G}_w + H_w R_{w-2k-1}^{w-1}.$$

Now, the proposition follows from straightforward computations. \square

By combining different cases in Proposition 4.2 we obtain the following useful results:

Corollary 4.3. *The monic orthogonal polynomials $\{R_w\}$ satisfy*

$$0 = x(M_w R_w(x) - R_w(x) M_w) + N_w R_w(x) + R_w(x) \tilde{N}_w,$$

where

$$M_w = \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix}, \quad N_w = - \begin{pmatrix} 0 & \frac{p(n-p+w)}{p+w} \\ \frac{(n-p)(p+w)}{n-p+w} & 0 \end{pmatrix}, \quad \tilde{N}_w = \begin{pmatrix} 0 & n-p \\ p & 0 \end{pmatrix}.$$

Proof. It follows from Proposition 4.2: combining the result for the values $a_{21} = 1, c_{12} = 0, a_{11} = 0, a_{22} = 0$ with the same result for the values $a_{21} = 2, c_{12} = 0, a_{11} = 0, a_{22} = 0$. \square

Corollary 4.4. *The orthogonal polynomials $\{R_w\}$ satisfy*

$$0 = x(M_w R_w(x) - R_w(x) M_w) + N_w R_w(x) + R_w(x) \tilde{N}_w + J_w R_{w-1},$$

where

$$M_w = \frac{(p+w)(n-p+w)}{n-2p} \begin{pmatrix} 0 & p \\ n-p & 0 \end{pmatrix}, \quad N_w = (w(w+n) - p(n-p)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\tilde{N}_w = \begin{pmatrix} p(n-p) & 0 \\ 0 & -w(w+n) \end{pmatrix}, \quad J_w = \frac{w(n+w)}{(n-1+2w)} \begin{pmatrix} 0 & \frac{p(n-p+w-1)}{p+w} \\ -\frac{(p+w-1)(n-p)}{(n-p+w)} & 0 \end{pmatrix}.$$

Proof. It follows from Proposition 4.2: combining the result for the values $a_{21} = 0, c_{12} = 2, a_{11} = 0, a_{22} = 0$ with the same result for the values $a_{21} = 0, c_{12} = 1, a_{11} = 0, a_{22} = 0$. \square

Starting with the differentiation formulas for the monic orthogonal polynomials R_w we obtain formulas corresponding to the orthonormal sequence Q_w .

Proposition 4.5. *Let $\{Q_w\}$ be an orthonormal sequence of matrix polynomials associated to the weight $W = W_{p,n}$ introduced in (4). We have*

$$(1 - x^2) \frac{dQ_w}{dx}(x) = -w x Q_w(x) + x(\overline{F}_w Q_w(x) - Q_w(x) F_w) + \overline{G}_w Q_w(x) + Q_w(x) \tilde{G}_w + \overline{H}_w Q_{w-1}(x),$$

with

$$\overline{F}_w = \|R_w\|^{-1} F_w \|R_w\|, \quad \overline{G}_w = \|R_w\|^{-1} G_w \|R_w\|, \quad \overline{H}_w = \|R_w\|^{-1} H_w \|R_{w-1}\|,$$

where F_w , G_w , H_w and \tilde{G}_w are those given in Proposition 4.2.

Proof. The statement follows by multiplying the identity given in Proposition 4.2 by $\|R_w\|^{-1}$ and from the fact that $Q_w = \|R_w\|^{-1} R_w$. \square

5. CONCLUSION AND OUTLOOK

The main result derived in the previous sections is the existence of an explicit differential operator \tilde{D} which, as we proved, commutes with S .

If one compares this result with the one in the celebrated series of papers by D. Slepian, H. Landau and H. Pollak one may say that we are at the stage of their first papers. What is needed now is an argument to conclude that the eigenfunctions of \tilde{D} will automatically be eigenfunctions of the integral operator S .

In the series of papers mentioned above the simplicity of the spectrum of \tilde{D} follows from classical Sturm-Liouville theory and this guarantees that they have found a good way to compute the eigenvectors of S .

In our situation, things could eventually be reduced to that case, but in principle \tilde{D} , as well as S , have “matrix valued eigenvalues,” and the appropriate notion of “simple spectrum” requires careful handling. For a recent careful analysis of the spectral problem in the scalar case, see [Kat16, KM12].

At this point this appears as a non-trivial project and we intend to develop it in a future publication. Part of this project is to develop numerical tools to compute the matrix valued eigenfunctions of \tilde{D} most likely using expansions of analogs of the Legendre polynomials used in [ORX13].

There are, of course, a number of different self-adjoint extensions of our symmetric differential operator \tilde{D} . We are in the “limit circle” situation of H. Weyl at both endpoints. Only one extension is of interest to us, this issue already appears in the scalar case, and it has been carefully discussed in [Kat16, KM12]. In our matrix value setup this will be part of the project mentioned above.

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